Trying to understand mass

B. Hoeneisen

November 16, 2006

Introduction

The origin of mass is a mystery. Particles may acquire mass, or change their mass, when they interact with a medium. Examples are photons interacting with free electrons in a plasma or in a metal, photons interacting with Cooper pairs in a superconductor, electrons propagating in the periodic potential of a crystal, and photons propagating in a waveguide.

The symmetries of the standard model Lagrangian prevent adding "by hand" mass terms for the fermions or bosons. These particles acquire mass due to their interaction with the Higgs field.

The limit $m \to 0$ is singular in the sense that phenomena present with $m \neq 0$ is absent with $m = 0$. Examples are the longitudinal polarization of W^+, W^- and Z, the coupling between the left and right Weyl components of the Dirac field, and the Cabibbo-Kobayashi-Maskawa matrix.

The origin of mass is the next frontier in high energy physics. In this talk I try to understand how particles acquire mass in general, and in particular, how they acquire mass in the standard model and beyond.

Dispersion relation

Let us consider the wave

$$\propto e^{i(\vec{k}\cdot\vec{r}-\omega t)} \tag{1}$$

with the dispersion relation

$$\omega_0^2 = \omega^2 - k^2c^2. \tag{2}$$

To every wave (1) there is associated a particle of energy

$$E = \hbar\omega \tag{3}$$

(Planck relation), momentum

$$\vec{p} = \hbar\vec{k} \tag{4}$$

(De Broglie relation), and velocity equal to the group velocity of the wave:

$$\vec{v} = \nabla_{\vec{k}}(\omega) = \frac{\vec{k}c^2}{\omega} = \frac{\vec{p}c^2}{E}. \tag{5}$$

We obtain

$$m^2c^4 = E^2 - p^2c^2, \tag{6}$$

where the “cut-off mass” is given by

$$mc^2 = \hbar\omega_0. \tag{7}$$

If the particle has zero mass,

$$E = pc, \qquad \omega = kc. \tag{8}$$

If the particle is massive, we obtain

$$E = \frac{mc^2}{\sqrt{1-(v/c)^2}}, \tag{9}$$

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-(v/c)^2}}. \tag{10}$$

Inertia

Waveguide with (x, y) increasing with z.

$$F \equiv \frac{dp_z}{dt} = \frac{d}{dt}\left(\frac{Ev_z}{c^2}\right) = \frac{E}{c^2}\frac{dv_z}{dt} \approx m\frac{dv_z}{dt} \quad (11)$$

if $v \ll c$. Therefore, the "inertial mass" is equal to the "cut-off mass".

In general, consider any system with energy E_0 in its rest frame, *i.e.* the frame with $\vec{p} = 0$. After a Lorentz transformation, the system has velocity $\vec{V}$ and momentum $\vec{p} = E_0\vec{V}/c^2$ if $V \ll c$. So the system has inertia with mass $m = E_0/c^2$. Inertia, and the conservation of energy-momentum, are two aspects of the same phenomenon.

Conservation of energy-momentum

Illuminate an atom with two beams of light:

$$A_1 e^{i(\vec{k_1}\cdot\vec{r_j}-\omega_1 t)} + A_2 e^{i(\vec{k_2}\cdot\vec{r_j}-\omega_2 t)}. \tag{12}$$

$$\sum_{n,m} a_{nm} e^{in(\vec{k_1}\cdot\vec{r_j}-\omega_1 t)} e^{im(\vec{k_2}\cdot\vec{r_j}-\omega_2 t)}, \tag{13}$$

with $n, m = 0, \pm 1, \pm 2 \cdots$. The electric field at the receiver is proportional to

$$\propto \sum_{n,m} a_{nm} e^{in(\vec{k}_1\cdot\vec{r}_j-\omega_1 t)} e^{im(\vec{k}_2\cdot\vec{r}_j-\omega_2 t)} e^{i\vec{k}_3\cdot(\vec{r}-\vec{r}_j)}. \tag{14}$$

Note that the amplitude at the receiver has components of frequency

$$\omega_3 = n\omega_1 + m\omega_2. \tag{15}$$

Multiplying by $\hbar$ (to get the conventional unit for energy) we obtain the equation of conservation of energy:

$$\hbar\omega_3 = n\hbar\omega_1 + m\hbar\omega_2. \tag{16}$$

In our example with positive n and m, n incoming photons of energy $E_1 = \hbar\omega_1$ scatter on m incoming photons of energy $E_2 = \hbar\omega_2$, producing an outgoing photon of energy $E_3 = \hbar\omega_3$. Note that *classical* waves interact in packets of frequency or energy.

It is convenient to include a factor $e^{i(\vec{k}_3\cdot\vec{r}-\omega_3 t)}$ in the proportionality constant of (14). Then equation (14) becomes

$$\propto \sum_{n,m} a_{nm} e^{in(\vec{k}_1\cdot\vec{r}_j-\omega_1 t)} e^{im(\vec{k}_2\cdot\vec{r}_j-\omega_2 t)} e^{-i(\vec{k}_3\cdot\vec{r}_j-\omega_3 t)}. \tag{17}$$

The part exhibited explicitly in (17) is independent of $(t, \vec{r})$.

Let us now consider a crystal. The amplitude at the receiver is a sum of terms (17) over all atoms j in the crystal:

$$\propto \sum_{j,n,m} a_{nm} e^{in(\vec{k}_1\cdot\vec{r}_j-\omega_1 t)} e^{im(\vec{k}_2\cdot\vec{r}_j-\omega_2 t)} e^{-i(\vec{k}_3\cdot\vec{r}_j-\omega_3 t)}. \tag{18}$$

Since, for a 3-dimensional crystal lattice, $\vec{r_j} = \vec{r_0} + N\vec{a} + M\vec{b} + L\vec{c}$ with N, M and L integers, it follows that the condition that the waves scattered at each atom j add up in phase at the receiver is

$$n\vec{k_1} + m\vec{k_2} = \vec{k_3} + \vec{G}, \qquad (19)$$

where $\vec{G} \cdot \vec{a}$, $\vec{G} \cdot \vec{b}$ and $\vec{G} \cdot \vec{c}$ are multiples of 2π. Equation (19) is the Bragg law, or, multiplying by $\hbar$, the law of conservation of "crystal momentum". In the limit of a continuous medium, i.e. in the limit $\vec{a}, \vec{b}, \vec{c} \to 0$, the only finite $\vec{G}$ is $\vec{G} = 0$, and we obtain the law of conservation of momentum.

Note that the Feynman rules to obtain the probability amplitude for a particular scattering is obtained by multiplying a factor $\exp\left[i(\vec{k_i} \cdot \vec{r_j} - \omega_i t)\right]$ for each incoming particle i, a factor $\exp\left[-i(\vec{k_o} \cdot \vec{r_j} - \omega_o t)\right]$ for each outgoing particle o, and a vertex factor proportional to a_{nm}.

The accuracy with which the conservation of energy-momentum is valid depends on the size of the interaction region and the time it takes for the interference to build up. The result of a "back-of-the envelope" calculation is the "uncertainty principle".

In conclusion, non-linear interactions of clasical *waves* produce *particles* with energy-momentum given by the Planck and De Broglie relations.

Quantum mechanics *chooses* among terms in the series.

Particle in a box

Consider a particle that collides elastically with the sides of a box. The box is accelerated in the z-direction by some external means. At time $t = 0$ the box is at rest (with respect to the inertial reference frame), and the particle collides elastically with one wall of the box. At time $t = \Delta t$ the particle collides with the opposite wall of the box, which is now moving in the z-direction with velocity $V = a \cdot \Delta t$. The change of p_z of the particle is less in the second collision than in the first collision due to the velocity V. The difference, calculated doing two Lorentz transformations, is

$$\Delta p = \frac{2V}{c^2} E \qquad (20)$$

if $V/c \ll 1$. The change of momentum per unit time is called "force". The time to go back and forth across the box is $2\Delta t$ if

$V/c \ll 1$. Therefore the force is given by Newton's equation

$$F = M \cdot a \tag{21}$$

with Einstein's famous inertial mass

$$M = \frac{E}{c^2}. \tag{22}$$

How does the proton acquire mass?

We try the following crude model: the proton is composed of three massless quarks in a box. For simplicity we take the box to be a cube of sides d with the same volume as a sphere of radius 1.2 fm. We place each of the three quarks in the lowest energy state with $\lambda/2 = d$. Then we obtain an inertial mass of the proton of order ≈ 0.96 GeV/c^2.

Excited states are predicted and observed.

Exploring the limit $m \to 0$

From now on we generally set $c = \hbar = 1$. A massive free field ϕ satisfies the wave equation

$$\partial_\mu \partial^\mu \phi + m^2 \phi = 0, \tag{23}$$

which is equivalent to the dispersion relation (2). This differential equation can be written in integral form as

$$\phi(x) = \phi_0(x) - \int S_0(x-y) m^2 \phi(y) d^4 y, \tag{24}$$

where $\phi_0(x)$ is the general solution of

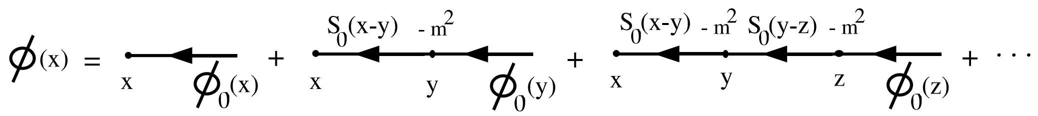
$$\partial_\mu \partial^\mu \phi_0 = 0, \tag{25}$$

and $S_0(x)$ is a particular solution of

$$\partial_\mu \partial^\mu S_0(x) = \delta^4(x). \tag{26}$$

Now we can treat m^2 as a perturbation, and solve (24) iteratively:

$$\phi(x) = \phi_0(x) + \int S_0(x-y)(-m^2)\phi_0(y)d^4y + \int S_0(x-y)(-m^2)S_0(y-z)(-m^2) \phi_0(z)d^4yd^4z + \ldots \quad (27)$$



Feynman diagrams of a massless scalar particle acquiring mass by forward scattering on the vacuum.

How does the electron acquire mass?

The Dirac field ψ has 4 components and carries a reducible representation of the proper Lorentz group. The irreducible components are Weyl-L and Weyl-R of dimension 2:

$$\psi = \psi_L \oplus \psi_R, \psi_L = \frac{1}{2}(1-\gamma^5)\psi, \psi_R = \frac{1}{2}(1+\gamma^5)\psi. \tag{28}$$

The Dirac equation for a free electron can be written as

$$\begin{aligned} -i\gamma^\mu \partial_\mu \psi_L &= -m\psi_R, \\ -i\gamma^\mu \partial_\mu \psi_R &= -m\psi_L, \end{aligned} \tag{29}$$

where γ^μ are the Dirac matrices.

The differential equations (29) can be written in integral form:

$$\begin{aligned} \psi_L(x) &= \psi_{L0}(x) + \int [-iS_0'(x-y)](-im)\psi_R(y)d^4y, \\ \psi_R(x) &= \psi_{R0}(x) + \int [-iS_0'(x-y)](-im)\psi_L(y)d^4y, \end{aligned} \tag{30}$$

where

$$-i\gamma^\mu \partial_\mu \psi_{L0} = 0, \qquad -i\gamma^\mu \partial_\mu \psi_{R0} = 0, \quad (31)$$

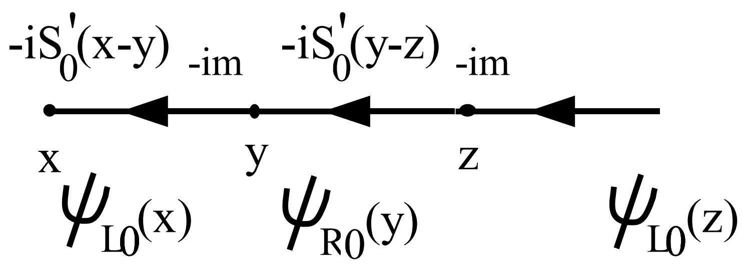
and

$$-i\gamma^\mu \partial_\mu S_0'(x) = \delta^4(x). \qquad (32)$$

Equations (30) can be iterated and we obtain the series

$$\begin{aligned}\psi_L(x) &= \psi_{L0}(x) + \int [-iS_0'(x-y)](-im)\psi_{R0}(y)d^4y \\ &+ \int [-iS_0'(x-y)](-im)[-iS_0'(y-z)] \\ &(-im)\psi_{L0}(z)d^4yd^4z + \cdots, \qquad (33)\end{aligned}$$

and a similar series for $\psi_R(x)$.



A Feynman diagram of a massless Dirac particle acquiring mass by forward scattering on the vacuum.

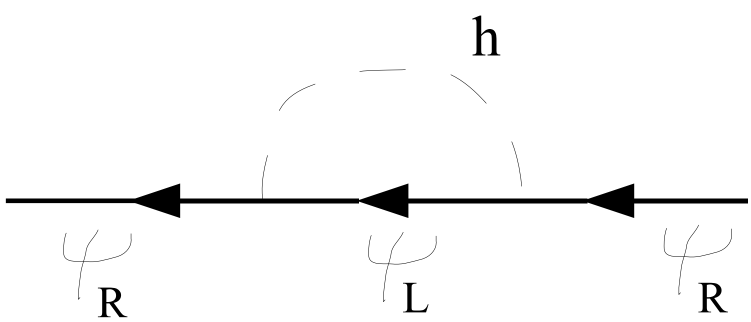
"Stepping stone model"

Weyl fields in a box

$$\propto \begin{pmatrix} \exp\{-\frac{\phi}{2}\} \exp\{ie^{-\phi}m(-z' - t')\} \\ 0 \\ \exp\{\frac{\phi}{2}\} \exp\{ie^{\phi}m(z' - t')\} \\ 0 \end{pmatrix} \quad (34)$$

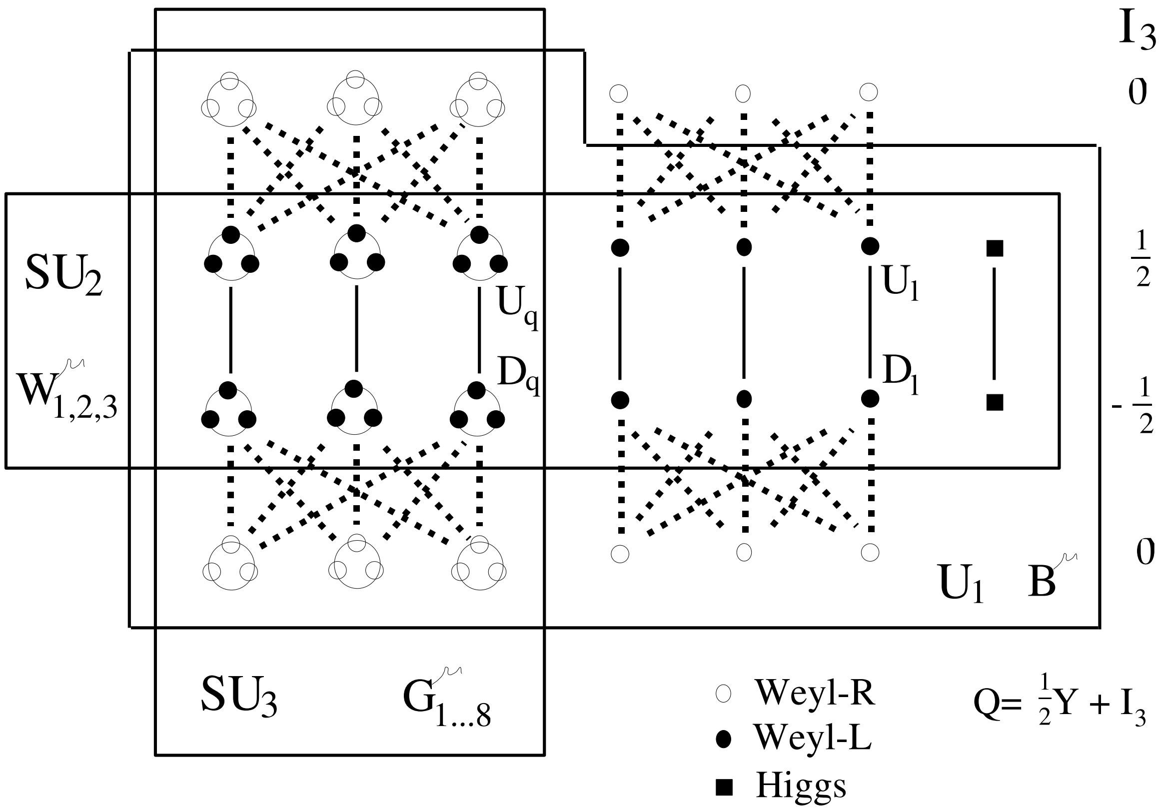
What is the box made of?

$$\propto \tilde{\psi}_L h \psi_R + \tilde{\psi}_R h \psi_L \qquad (35)$$



Feynman diagram of an electron in a "box".

The standard model



The elementary particles of the standard model.

Mass revisited

The term in the standard model Lagrangian that gives mass to the electron is

$$-G_e \frac{v+h}{\sqrt{2}} \left(\tilde{\psi}_R \psi_L + \tilde{\psi}_L \psi_R \right), \qquad (36)$$

where G_e is a dimensionless "Yukawa coupling" put in by hand. So the vacuum expectation value v of the Higgs field gives the electron a "tree level" mass $m_e = G_e v/\sqrt{2}$. Note in (36) that the Lagrangian indeed contains a term (35). So the standard model contains loops as shown in the Figure. Summing a series with 0, 1, 2... loops we obtain a propagator with the renormalized mass

$$m_e = \frac{G_e}{\sqrt{2}} \left(v + \frac{G_e}{\sqrt{2}} A + \cdots \right), \qquad (37)$$

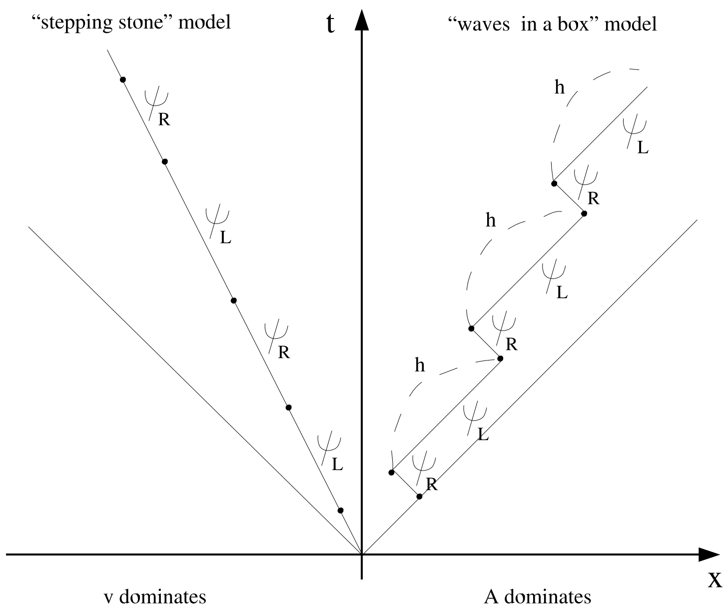
where the factor A has the form

$$A = -i \int \frac{d^4k}{(2\pi)^4} \frac{i}{k_\mu k^\mu - m_h^2} \frac{i}{\gamma^\mu (p_\mu + k_\mu) - \frac{G_e v}{\sqrt{2}}}. \qquad (38)$$

We then have two alternatives. In the first alternative, the term with v dominates in (37), and the electron acquires mass by forward scattering on the vacuum. This is the "stepping stone" model.

In the second alternative, the terms $\frac{G_e}{\sqrt{2}}A + \cdots$ dominate, and we obtain the model of Weyl-L and Weyl-R fields reflecting back and forth in a box made of the Higgs field. This is the "waves in a box" model.

Which alternative has nature chosen? Or has nature chosen a combination of both?



Two alternatives to acquire mass.

Experimental consequences

The proton can be viewed as ultrarelativistic quarks in a box made of gluons. Note that the proton acquires mass independently of the vacuum expectation value of the Higgs field, and this mass can be calculated with lattice quantum chromodynamics.

The case of the electron is more uncertain. Apparently, the electron acquires mass by the "stepping stone" mechanism. If the electron can be considered as "waves in a box", then the box has size $\approx \hbar/(mc)$, the field h would have to be nearly massless, and the size of the box would cause the Darwin shift of the energy levels of hydrogen.

Conclusions

Nature, as we currently understand it, is described by a perturbation expansion of Feynman diagrams. We subscribe to the belief that “the diagrams contain more truth than the underlying formalism” [’t Hooft and Veltman (1973)].

We can not calculate the electron mass: we must introduce a “Yukawa coupling” by hand *at each order in the perturbation expansion*. On the other hand, we can calculate the mass of baryons and mesons using lattice quantum chromodynamics.

We considered *classical* waves that interact. We found that the condition that the waves add up in phase at the receiver has the form

$$\begin{aligned} n\omega_1 + m\omega_2 + \cdots &= 0, \\ n\vec{k}_1 + m\vec{k}_2 + \cdots &= 0, \end{aligned} \tag{39}$$

where the powers $n, m \cdots$ of the Taylor expansion of the non-linear interaction, are integers. Note that the concept of particle, and the Planck and De Broglie relations, have emerged from the interaction of *classical* waves!

In semiconductor physics we can consider quasi-particles called "electrons" and "holes" with their effective masses, or, at a deeper level, we can consider electrons being Bragg-reflected back and forth by the periodic potential of the crystal. Similarly, we can consider the massive Dirac electron, or, at a deeper level, we can perhaps consider massless Weyl-L and Weyl-R fields being reflected back and forth by emitting and absorbing virtual scalar Higgs particles. Similarly, we can perhaps consider massive gauge bosons as massless bosons being reflected back and forth by emitting and absorbing virtual scalar Higgs particles. These reflections give the gauge bosons their longitudinal polarization (as reflections in a waveguide give the photon a mass and a longitudinal polarization). Which model best describes nature, "waves in a box" or "stepping stone", or a combination of both, will depend on which term in (37) dominates.

In the process of trying to understand mass, I have arrived at the following view of the standard model. Quarks and leptons are not elementary at all. An electron is a messy composite of a superposition of three Weyl-L fields, a Weyl-R field and a Higgs field coupled together by a Yukawa coupling, and all of them are coupled to gauge fields by gauge couplings, which also interact with themselves and everything else. The energy of this mess has inertia, as all energy does, and it is that mess that determines the mass of the electron. So the next level in our understanding of Nature, from atoms to nuclei and electrons, to baryons and mesons, to quarks and leptons, are the dots shown in the Figure, and these have no mass (except the Higgs?).

Many questions remain. The reason for an effective, low energy, theory to be renormalizable is to decouple it from the high energy theory. But, in the standard model, the Higgs sector does not decouple. What are the consequences on e^+e^- scattering of Feynman graphs with Higgs in a loop? If new physics cuts off otherwise divergent integrals such as A, what are the experimental consequences? Can we constrain new physics this way?